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Source-Type Solutions to Thin-Film Equations: The Critical Case

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Abstract—We prove that the limiting behaviour of the source-type solutions for the equation $h_t + \operatorname{div}(h^n \operatorname{grad}(\Delta h)) = 0$ when $n \to 3$ is a δ function which remains fixed. A rescaled limiting profile is also obtained.

Keywords—Thin-film flows, Source solutions, Limiting profile.

1. INTRODUCTION

In this note, we study the critical limit problem $n = 3$ of the existence of source-type solutions of the equation

$$h_t + \operatorname{div}\,(h^n \operatorname{grad}\,(\Delta h)) = 0, \qquad \text{in } \mathbb{R}^d \times (0, \infty), \tag{1.1}$$

i.e., the existence of nonnegative solutions of (1.1) which satisfy

$$h(\cdot, t) \to M\delta, \qquad \text{as } t \to 0^+, \tag{1.2}$$

where δ is the Dirac mass and n and M are positive constants. Moreover, we shall require the point-wise convergence

$$h(x,t) \to 0, \quad \text{as } t \to 0^+, \qquad \text{for all } x \neq 0.$$

These source solutions were studied in [1] for $d = 1$ and in [2] for $d \geq 2$. Equation (1.1) arises in several applications. The case $n = 3$ appears in the lubrication theory for thin viscous films that are driven by surface tension, where h is the height of the film. For $n = 1$, it models the flow of a thin neck of fluid of width $2h$ in a Hele-Shaw cell. For an overview on degenerate parabolic equations of higher order and their applications, we refer to [3]. The critical character of the value $n = 3$ appears in several aspects of the theory, see [2] and the references therein.

As in [2], we search the source solutions with the radial, self-similar form,

$$h(x,t) = t^{-d\beta} f(r), \quad r = \frac{|x|}{t^\beta}, \qquad \beta = \frac{1}{(4 + dn)}. \tag{1.3}$$

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The function $f(r)$ has bounded support $[0,a]$ and is the solution of the following problem:

$$\begin{gathered}\text{find a number } a>0 \text{ and a function } f \text{ such that}\\ f\in C^3[0,a)\cap C^1[0,a], \qquad f>0 \text{ in } [0,a),\\ (\Delta_r f)'=\beta r f^{1-n}, \qquad \text{in } (0,a),\\ f'(0)=f(a)=f'(a)=0, \qquad \omega_d\int_0^a r^{d-1}f(r)\,dr=M,\end{gathered} \tag{1.4}$$

where Δ_r is the radial Laplacian and ω_d the area of the unit sphere in $\mathbb{R}^d$. Setting $u(r)=cf(ar)$ with $c^n a^4\beta=1$, we find that $\operatorname{supp}(u)=[0,1]$ and $u(r)$ is a solution of

$$\begin{gathered} u\in C^3[0,1)\cap C^1[0,1], \qquad u>0, \quad \text{in } [0,1),\\ (\Delta_r u)'=ru^{1-n}, \qquad \text{in } (0,1),\\ u'(0)=u(1)=u'(1)=0.\end{gathered} \tag{1.5}$$

Conversely, let $u(r)$ be a solution of (1.5); if we define a and c by

$$c^n a^4\beta=1, \qquad cM=\omega_d a^d\int_0^1 r^{d-1}u(r)\,dr, \tag{1.6}$$

then $f(r)=(1/c)u(r/a)$ is a solution of (1.4).

The purpose of this note is to investigate the behaviour of these source-type solutions as n approaches three. We shall prove the following theorem.

THEOREM A. *Let f_n and a_n be the function and number, respectively, that solve system (1.4) with exponent n. These behave as follows.*

(1) $f_n(0)\to\infty$ *as* $n\to 3$.
(2) $a_n\to 0$ *as* $n\to 3$.

Let $h_n(x,t)$ be the corresponding source-type solution. Then we have the following.

(1) $h_n(x,t)\to M\delta(x)$ *for all time* $t>0$.
(2) *The rescaled function*

$$H_n(x,t)=a_n^d\,h_n(a_nx,t)\to\frac{d(d+2)M}{2\omega_d}t^{-d\beta}\left(1-|x|^2t^{-2\beta}\right)_+, \qquad \text{as } n\to 3.$$

Notice that the limit of H_n is, for each t, a parabolic arch with nonzero contact angle.

In what follows, the same letter C will denote different positive constants, all of which will be independent of n.

2. PRELIMINARIES

In [2], we studied system (1.5) and proved the following assertions.

(1) There exists a unique solution of (1.5) if $0<n<3$ and there is no solution for $n\geq 3$. Moreover, this solution satisfies

$$u^{(j)}(r)=\int_0^1 G^{(j)}(r,t)tu^{1-n}(t)\,dt, \qquad j=0,1, \tag{2.1}$$

where $G(r,t)$ is the Green function associated to system (1.5) as explained in [2].

(2) The Green function verifies the following estimates: $G(r,t)>0$, $G'(r,t)<0$ and for all $r,t\in[0,1]$,

$$G(r,t)\leq C(1-t) \quad\text{and}\quad |G'(r,t)|\leq C(1-t). \tag{2.2}$$

(3) The Maximum Principle (Lemma 4.1 of [2]) implies that

$$u > 0 \text{ in } [0,1), \quad u' < 0 \text{ in } (0,1), \qquad u''\left(0^+\right) < 0 \quad \text{and} \quad u''\left(1^-\right) > 0. \tag{2.3}$$

(4) The solution $u(r)$ satisfies that for $1 < n < 3$,

$$u_n(r) \geq C^{1/n}(1-r)^{3/n}. \tag{2.4}$$

(5) If $2/3 < n < 3$, then

$$u(r) \sim \left(\frac{n^3}{3(3-n)(2n-3)}\right)^{1/n} (1-r)^{3/n}, \qquad \text{as } r \to 1. \tag{2.5}$$

3. PASSAGE TO THE LIMIT IN THE PROBLEM (1.5)

LEMMA 3.1. *Let $q(t)$ be a function in $C^3[0,1) \cap C^1[0,1]$ for which*

$$q'(0) = q(1) = q'(1) = 0, \qquad \lim_{t\to 1}(1-t)\Delta_r q(t) = 0.$$

Then $q(r)$ satisfies the identity

$$\int_0^1 t^d \left(1-t^2\right) \left(\Delta_r q(t)\right)' \, dt = 2d(d+2) \int_0^1 t^{d-1} q(t) \, dt.$$

PROOF. The identity is obtained using integration by parts. ∎

LEMMA 3.2. *Take u_n to be the solution to problem (1.5) with exponent n. Then*

$$\int_0^1 r^{d-1} u_n(r) \, dr \to \infty, \qquad \text{as } n \to 3.$$

PROOF. In order to obtain a contradiction, let us assume that there exists a bounded subsequence $\{u_n\}$ such that

$$\int_0^1 r^{d-1} u_n(r) \, dr \leq C.$$

By (2.5), we have that

$$(1-r)\Delta_r u_n(r) \to 0, \qquad \text{as } r \to 1.$$

This, together with the boundary conditions, gives that u_n satisfies the hypotheses of Lemma 3.1. Hence, setting

$$I_n = \int_0^1 t^d \left(1-t^2\right) \left(\Delta_r u_n\right)'(t) \, dt,$$

we obtain

$$I_n \leq C. \tag{3.1}$$

On the other hand, by (2.1) and (2.2), we have the following estimate:

$$\begin{aligned} \left|u_n'(r)\right| &= \left|\int_0^1 G'(r,t)(\Delta_r u_n)'(t) \, dt\right| \\ &\leq C \int_0^{1/2} (1-t) \left(\Delta_r u_n\right)'(t) \, dt + C \int_{1/2}^1 (1-t) \left(\Delta_r u_n\right)'(t) \, dt. \end{aligned}$$

By (3.1), the second integral is bounded. For the first integral, we trivially have

$$\int_0^{1/2} (1-t)\left(\Delta_r u_n\right)'(t)\,dt = \int_0^{1/2} (1-t) t u_n^{1-n}(t)\,dt \le C u_n^{1-n}\left(\frac{1}{2}\right),$$

where the last inequality is given by the fact that u_n is a decreasing function. Now, by (2.4), the first integral is bounded too.

Having shown that $|u_n'(r)|$ is bounded, integration between r and 1 gives that $u_n \le C(1-r)$. From this, it follows that

$$I_n \ge \int_0^1 \frac{t^{d+1}\left(1-t^2\right)}{C^{n-1}(1-t)^{n-1}} \to \infty, \qquad \text{as } n \to 3.$$

This is a contradiction. ∎

In order to study the limit profile, we define the function

$$H_n(x,t) = a_n^d h_n\left(a_n x, t\right),$$

which verifies the property of mass conservation. Writing H_n in terms of the function u_n, we have

$$H_n(x,t) = t^{-d\beta} F_n(r), \qquad r = \frac{|x|}{t^\beta},$$

where by (1.3) and (1.6)

$$F_n(r) = \frac{u_n(r)}{Q_n}, \qquad Q_n = \frac{\omega_d \int_0^1 r^{d-1} u_n(r)\,dr}{M}.$$

Hence, $\operatorname{supp}(F_n) = [0,1]$. Notice that $Q_n \to \infty$ as $n \to 3$. On the other hand, by (2.1),(2.2), we have that for $j = 0, 1$,

$$\left|F^{(j)}\right| \le \frac{C}{Q_n}\left(\int_0^{1/2} (1-t)\left(\Delta_r u_n\right)'(t)\,dt + \int_{1/2}^1 (1-t)\left(\Delta_r u_n\right)'(t)\,dt\right).$$

The first integral is bounded as before. For the second integral, we use the fact that u_n satisfies Lemma 3.1. Then

$$\int_{1/2}^1 (1-t)\left(\Delta_r u_n\right)'(t)\,dt \le \frac{2^{d+2} d(d+2)}{3} \int_0^1 r^{d-1} u_n(r)\,dr.$$

Hence, the functions F_n and F_n' are uniformly bounded in n, and thus, there exists a subsequence such that $F_n \to F$ uniformly in $[0,1]$, with $F \ge 0$ and $F' \le 0$ in $[0,1]$.

LEMMA 3.3. *The function $F(r)$ satisfies*

$$\begin{aligned} F(r) &> 0, && \text{in } [0,1), \\ \left(\Delta_r F\right)' &= 0, && \text{in } (0,1), \\ F(1) &= 0, && \omega_d \int_0^1 r^{d-1} F(r)\,dr = M. \end{aligned}$$

Therefore,

$$F(r) = \frac{d(d+2)M}{2\omega_d}\left(1-r^2\right)_+. \tag{3.2}$$

PROOF. By the uniform convergence of F_n, $F(1) = 0$, $\omega_d \int_0^1 r^{d-1} F(r)\, dr = M$, and $(\Delta_r F_n)' \to (\Delta_r F)'$ in distributional sense. Moreover, $(\Delta_r F_n)' \geq 0$. Then the distribution $(\Delta_r F)'$ is nonnegative. Hence, $(\Delta_r F)'$ is a measure, $\Delta_r F$ is locally of bounded variation, and $F \in C^1(0,1)$.

Let us suppose that there exists $b \in (0,1)$ such that $F \equiv 0$ in $[b,1]$. Uniform convergence, together with the fact that $F > 0$ in $[0,b)$, gives that

$$(\Delta_r F_n)' = \frac{r F_n^{1-n}}{Q_n^n} \to 0,$$

uniformly in compact sets of $[0,b)$. Therefore, in $(0,b)$, the function F satisfies the equation

$$(\Delta_r F)' = 0.$$

Then

$$F(r) = C_1 + C_2 r^2 + C_3 \phi(r),$$

with $\phi(r) = r^{2-d}$ if $d > 2$ and $\phi(r) = \log(r)$ if $d = 2$. On the other hand, since F is bounded, $C_3 = 0$; $F'(b) = 0$, $C_2 = 0$, and $F(b) = 0$, $C_1 = 0$. This contradiction proves that the function $F > 0$ in $[0,1)$. Repeating the same argument, one obtains that for $r \in [0,1]$,

$$F(r) = C_1 \left(1 - r^2\right),$$

and by the mass conservation, $C_1 = (d(d+2)M)/(2\omega_d)$. ∎

REMARK. Lemma 3.3 is also true for $d = 1$. In this case, $F'(0) = 0$ follows easily because there is no singularity at $r = 0$.

4. PROOF OF THEOREM A

By the definition of problem (1.4), the profile $f_n(r)$ satisfies

$$\omega_d \int_0^{a_n} r^{d-1} f_n(r)\, dr = M. \tag{4.1}$$

Hence, by (1.6), we find that

$$M = \omega_d \beta^{1/n} a_n^{d+4/n} \int_0^1 r^{d-1} u_n(r)\, dr,$$

and it follows from Lemma 3.2 that $a_n \to 0$ as $n \to 3$.

From (4.1), and the fact that $f_n(r) \leq f_n(0)$, we have

$$M \leq \omega_d f_n(0) \frac{a_n^d}{d}.$$

As a result, $f_n(0) \to \infty$ as $n \to 3$. Hence, the two first points of Theorem A are proved.

Now, we consider the source type solution h_n given by (1.3); we have that for all fixed $t > 0$, this function satisfies that as $n \to 3$,

$$\begin{gathered} h_n(0,t) = t^{-d\beta_n} f_n(0) \to \infty, \\ \operatorname{supp}(h_n(x,t)) = \left\{|x| \leq a_n t^{\beta_n}\right\} \to \{0\}, \\ \omega_d \int_{\mathbb{R}^d} h_n(x,t)\, dx = M. \end{gathered}$$

Therefore, $h_n(\cdot,t) \to M\delta$ as $n \to 3$.

On the other hand, from the definition of $H_n(r)$ and due to Lemma 3.3, we have that

$$H_n(x,t) = a_n^d\, h_n(a_n x, t) \to \frac{d(d+2)M}{2\omega_d} t^{-d\beta} \left(1 - |x|^2 t^{-2\beta}\right)_+, \qquad \text{as } n \to 3.$$

This completes the proof of Theorem A.

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